An Extension of Gordan's Lemma for Infinite Systems of Strict Linear Inequalities

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1 Introduction

Let S be an arbitrary subset of $\mathbb{R}^{n+1}$. The general system of strict linear inequalities that we wish to consider in this paper has the following form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n > b \quad \text{for all } (a_1, a_2, \ldots, a_n, b) \in S. \tag{1}$$

It is well-known that, in case S is a finite set, Gordan's lemma gives a necessary and sufficient condition for the existence of solutions to (1): *there exist real-valued solutions* $x_1, x_2, \ldots, x_n$ *to* (1) *if and only if* $\mathbf{0}$ *is not a linear convex combination of* $S \cup \{(0, \ldots, 0, -1)\}$ (where $\mathbf{0}$ denotes the origin of $\mathbb{R}^{n+1}$).

Gordan's lemma cannot however be directly extended for the infinite case, as shown in the following example: $S' = \{(1,1), (1,2), (1,3), \ldots, (1,k), \ldots\}$. One can easily verifies that $(0,0)$ is not a convex combination of $S' \cup \{(0,-1)\}$, but the corresponding system of strict linear inequalities has no real-valued solution:

$$x > k \quad \text{for all positive integer } k. \tag{2}$$

Let ω be an imaginary solution to (2), which we call an *infinitely large number*. The aim of this paper is to show that Gordan's lemma can be extended for the infinite case if we adopt such an infinitely large number ω as a solution to the system of strict linear inequalities.

For this purpose we introduce an extended structure of the field of real numbers $\mathbb{R}$ with an infinitely large number ω, showing that this structure coincides with the set of lexicographically ordered vectors.

As Gordan's lemma is proved by way of the usual separating hyperplane lemma, the main result of this paper is derived as a consequence of the lexicographical separation theorem of Hausner and Wendel [6], Klee [8], Martínez-Legaz and Singer [11] that any two disjoint convex sets in $\mathbb{R}^n$ can be separated lexicographically.

In Section 2 we review some basic results on the lexicographic order on $\mathbb{R}^n$. Also we introduce a new concept called "lexicographically ordered polynomials," showing that the lexicographic order can be described by the ring of Laurant polynomials whose variable is infinitely large. The concept of lexicographically ordered polynomials is a slight modification of that proposed by Hammond [4].

In Section 3 we define some fundamental notions in convexity theory. Then we present the lexicographical separation theorem mentioned above, and modify it in terms of lexicographically ordered polynomials.

In Section 4 we provide our main results of this paper, where we apply the lexicographical separation theorem to strict linear inequality systems. Our goal is prove Theorem 4.2, giving a necessary and sufficient condition for the existence of solutions to infinite systems of strict linear inequalities. Here we allow the solutions to be lexicographically ordered polynomials. Several examples of strict linear inequality systems exhibit that it is not unreasonable to obtain such lexicographically ordered polynomials in our solutions.

In Section 5 we give concluding remarks.

2 Lexicographic Orders

In this paper, the elements of $\mathbb{R}^n$ will be considered as column vectors, and the superscript ^{T} will mean transpose. First, we introduce the lexicographic order on $\mathbb{R}^n$.

Definition 2.1 Let $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, $\boldsymbol{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$. We define

$$\boldsymbol{x} <_L \boldsymbol{y} \quad \text{iff} \quad \boldsymbol{x} \neq \boldsymbol{y} \quad \text{and} \quad x_k < y_k \ \text{ for } k = \min\{ i \mid x_i \neq y_i \}, \tag{3}$$

where $<_L$ denotes "lexicographically less than." We also write $\boldsymbol{x} \leq_L \boldsymbol{y}$ to mean $\boldsymbol{x} <_L \boldsymbol{y}$ or $\boldsymbol{x} = \boldsymbol{y}$. □

The next proposition is easy to check, so we omit its proof.

Proposition 2.1 Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$.

(a) $\boldsymbol{a} <_L \boldsymbol{b}$ and $\boldsymbol{c} \leq_L \boldsymbol{d}$ imply $\boldsymbol{a}+\boldsymbol{c} <_L \boldsymbol{b}+\boldsymbol{d}$,

(b) $\boldsymbol{a} <_L \boldsymbol{b}$ and $\lambda > 0$ imply $\lambda \boldsymbol{a} <_L \lambda \boldsymbol{b}$. □

We say that a matrix M is *lower triangular* if all entries above the main diagonal are zero; and *unitary* if all the diagonal elements are equal to one. The next proposition states that the lexicographic order is preserved under a unitary lower triangular transformation:

Proposition 2.2 Let $\boldsymbol{a} = (a_1, \ldots, a_n)^T$, $\boldsymbol{0} = (0, \ldots, 0)^T \in \mathbb{R}^n$. For all $n \times n$ unitary lower triangular matrices M, $\boldsymbol{a} >_L \boldsymbol{0}$ implies $M\boldsymbol{a} >_L \boldsymbol{0}$.

Proof. This is proved by induction on n. For $n = 1$ the statement is obvious. For $n > 2$, we assume that it is true for $n-1$. Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)^T >_L \boldsymbol{0}$, and let M be an $n \times n$ unitary lower triangular matrix. Let $\tilde{\boldsymbol{a}} = (a_2, \ldots, a_n)^T$. Since M can be written as $\begin{pmatrix} 1 & 0 \\ * & \tilde{M} \end{pmatrix}$ for some $(n-1) \times (n-1)$ unitary lower triangular matrix $\tilde{M}$, we have

$$\begin{aligned} & \boldsymbol{a} >_L \boldsymbol{0} \\ \Longleftrightarrow\ & a_1 > 0 \text{ or } \text{“} a_1 = 0 \text{ and } \tilde{\boldsymbol{a}} >_L \boldsymbol{0} \text{”} \\ \Longrightarrow\ & a_1 > 0 \text{ or } \text{“} a_1 = 0 \text{ and } \tilde{M}\tilde{\boldsymbol{a}} >_L \boldsymbol{0} \text{”} \ \text{(by the induction hypothesis)} \\ \Longleftrightarrow\ & M\boldsymbol{a} >_L \boldsymbol{0}. \end{aligned}$$

Thus we get the conclusion. □

Corollary 2.1 Let $\boldsymbol{a} = (a_1, \ldots, a_n)^T$, $\boldsymbol{b} = (b_1, \ldots, b_n)^T \in \mathbb{R}^n$. For all unitary lower triangular matrices M, $\boldsymbol{a} >_L \boldsymbol{b}$ implies $M\boldsymbol{a} >_L M\boldsymbol{b}$.

Proof. Suppose $\boldsymbol{a} >_L \boldsymbol{b}$. Then we have $\boldsymbol{a} - \boldsymbol{b} >_L \boldsymbol{0}$ by Proposition 2.1. Hence $M\boldsymbol{a} - M\boldsymbol{b} >_L \boldsymbol{0}$ by Proposition 2.2, and therefore $M\boldsymbol{a} >_L M\boldsymbol{b}$ by Proposition 2.1. □

Proposition 2.2 can also be derived directly from the statement below Theorem 3.6 of Martínez-Legaz [10], which shows that $A\boldsymbol{x} >_L \boldsymbol{0}$ for every $\boldsymbol{x} >_L \boldsymbol{0}$ if and only if A is lower triangular and all of its diagonal entries are strictly positive. See also Martínez-Legaz [9] and Martínez-Legaz and Singer [12] for some related topics on the lexicographic order in terms of linear operators on $\mathbb{R}^n$.

A *Laurant polynomial* is a polynomial in t and t^{-1} of the form

$$\sum_{-m}^{n} a_i t^i = a_n t^n + \cdots + a_1 t + a_0 + a_{-1} t^{-1} + \cdots + a_{-m} t^{-m}. \tag{4}$$

More precisely, *the ring* $\mathbb{R}[t,t^{-1}]$ *of Laurant polynomials in t over* $\mathbb{R}$ is constructed as the quotient ring $\mathbb{R}[t,x]/(xt-1)$ identifying x with t^{-1}, where $\mathbb{R}[t,x]$ is the polynomial ring in two variables t and x over $\mathbb{R}$, and $(xt-1)$ denotes the principal ideal generated by $xt-1$. (See e.g. Artin [1] for more details.) It is not difficult to verify that $t \cdot t^{-1} = 1$ and that all the laws for rings are satisfied.

Let us suppose that there is an order $<$ on $\mathbb{R}[t,t^{-1}]$. Consider the following conditions:

A1 $<$ is a total order on $\mathbb{R}[t,t^{-1}]$.
A2 for all $x, y, x', y', z \in \mathbb{R}[t,t^{-1}]$,
  (a) $x < y$ and $x' < y'$ imply $x + x' < y + y'$,
  (b) $x < y$ and $z > 0$ imply $x \cdot z < y \cdot z$,
A3 the restriction of $<$ to the subring $\mathbb{R}$ is the usual strict order on the real numbers,
A4 $t > k$ for all positive integer k.

Note that A1 and A2 are the standard conditions of *ordered rings*, and A3 means that the order $<$ is compatible with the ordering of $\mathbb{R}$. A4 is the statement that the monomial t is *infinitely large* in comparison with $\mathbb{R}$. As we shall see in the following lemmas, these conditions A1–A4 give a characterization of the lexicographic order; we shall show that, if t is infinitely large, the order $<$ on $\mathbb{R}[t,t^{-1}]$ is uniquely determined as the lexicographic order (Lemma 2.3).

In the following we write $x \le y$ to mean $x < y$ or $x = y$.

Lemma 2.1 Let $<$ be a relation on $\mathbb{R}[t,t^{-1}]$ satisfying A1–A4. Then, for all positive integer m,

(a) $t^m > k$ for all positive integer k.
(b) $0 < t^{-m}$ and $t^{-m} < 1/k$ for all positive integer k.

Proof. (a) This is proved by induction on m. For $m = 1$ the statement is trivial. For $m > 1$, we assume that $t^{m-1} > k$ for all positive integer k. By multiplying $t\,(>0)$, we have $t^m = t^{m-1} \cdot t > kt$ by A2. Since $t > 1$ by A4, we also have $kt > k$. Therefore $t^m > kt > k$.

(b) Suppose $t^{-m} \leq 0$. Since $t^m > 0$ by (a), we have $1 = t^m \cdot t^{-m} \leq 0$ by A2, a contradiction. Thus $t^{-m} > 0$. Next we show that $t^{-m} < 1/k$ for all positive integer k. Suppose othewise: $t^{-m} \geq 1/k$ for some k. Then, multiplying $kt^m (> 0)$ to both sides, we have $k = kt^m \cdot t^{-m} \geq t^m$ by A2, contradicting (a). □

According to Lemma 2.1, we shall say that t^m is *infinitely large*, and t^{-m} is *infinitely small* in comparison with $\mathbb{R}$.

Now, let $\mathbb{R}[t^{-1}]$ be the ring of polynomials in t^{-1}, i.e.

$$\mathbb{R}[t^{-1}] = \{a_0 + a_1 t^{-1} + \cdots + a_n t^{-n} \mid n \in \mathbb{N}, {}^{*1}\, a_0, a_1, \ldots, a_n \in \mathbb{R}\}. \tag{5}$$

Note that $\mathbb{R}[t^{-1}]$ is a subring of $\mathbb{R}[t, t^{-1}]$.

Lemma 2.2 Let $<$ be a relation on $\mathbb{R}[t, t^{-1}]$ satisfying A1–A4. Then, for all $n \in \mathbb{N}$ and all $a_0, a_1, \ldots, a_n \in \mathbb{R}$,

$$a_0 + a_1 t^{-1} + \cdots + a_n t^{-n} > 0 \quad \text{iff} \quad (a_0, a_1, \ldots, a_n)^T >_L \mathbf{0}. \tag{6}$$

Proof. It is easy to check that (6) is equivalent to the following two conditions:

$$\text{if } a_k > 0 \text{ for } k = \min\{i \mid a_i \neq 0\}, \text{ then } a_0 + a_1 t^{-1} + \cdots + a_n t^{-n} > 0, \tag{7}$$

$$\text{if } a_k < 0 \text{ for } k = \min\{i \mid a_i \neq 0\}, \text{ then } a_0 + a_1 t^{-1} + \cdots + a_n t^{-n} < 0. \tag{8}$$

Hence, it is enough to show (7); one can easily derive (8) in a parallel way with (7).

Suppose $p(t) = a_0 + a_1 t^{-1} + \cdots + a_n t^{-n}$ is a nonzero polynomial with $a_k > 0$ for $k = \min\{i \mid a_i \neq 0\}$. We shall show $p(t) > 0$.

First we consider the case $k = 0$, i.e. $a_0 > 0$. If also $a_i \geq 0$ for all $i > 0$, then $p(t) > 0$ is an immediate consequence of Lemma 2.1 (b). Suppose $a_i < 0$ for some $i > 0$, and let $a = \max\{|a_i| \mid a_i < 0\}$. Then, there is a sufficiently large integer M such that $0 < 1/M < a_0/na$. Thus, by Lemma 2.1 (b), we get

$$p(t) = a_0 + a_1 t^{-1} + \cdots + a_n t^{-n} > a_0 - a(t^{-1} + \cdots + t^{-n}) > a_0 - na/M > 0.$$

For the case $k > 0$, let $q(t)$ be the polynomial $q(t) = a_k + a_{k+1} t^{-1} + \cdots + a_n t^{-(n-k)}$ such that $p(t) = q(t) \cdot t^{-k}$. Then $a_k > 0$ and hence, applying the above argument to $q(t)$, we have $q(t) > 0$. Therefore, with $t^{-k} > 0$, we obtain $p(t) = q(t) \cdot t^{-k} > 0$. □

Lemma 2.3 Let $<$ be a relation on $\mathbb{R}[t, t^{-1}]$. Suppose $<$ satisfies A1–A4. Then, for all $n, m \in \mathbb{N}$ and all $a_n, \ldots, a_1, a_0, a_{-1}, \ldots, a_{-m}, b_n, \ldots, b_1, b_0, b_{-1}, \ldots, b_{-m} \in \mathbb{R}$,

$$\begin{aligned} &(a_n, \ldots, a_1, a_0, a_{-1}, \ldots, a_{-m})^T <_L (b_n, \ldots, b_1, b_0, b_{-1}, \ldots, b_{-m})^T \quad \text{iff} \\ &a_n t^n + \cdots + a_1 t + a_0 + a_{-1} t^{-1} + \cdots + a_{-m} t^{-m} \\ &\qquad < b_n t^n + \cdots + b_1 t + b_0 + b_{-1} t^{-1} + \cdots + b_{-m} t^{-m}. \end{aligned} \tag{9}$$

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*1 $\mathbb{N}$ denotes the set of natural numbers including zero.

Proof. The following holds for all $a_n, a_{n-1} \dots, a_{-m}, b_n, b_{n-1} \dots, b_{-m} \in \mathbb{R}$:

$$\begin{aligned}
& (a_n, a_{n-1}, \dots, a_{-m})^T <_L (b_n, b_{n-1}, \dots, b_{-m})^T \\
\iff & (b_n - a_n, b_{n-1} - a_{n-1}, \dots, b_{-m} - a_{-m})^T >_L 0 \quad \text{(by Proposition 2.1)} \\
\iff & (b_n - a_n) + (b_{n-1} - a_{n-1})t^{-1} + \dots + (b_{-m} - a_{-m})t^{-(n+m)} > 0 \quad \text{(by Lemma 2.2)} \\
\iff & [(b_n - a_n) + (b_{n-1} - a_{n-1})t^{-1} + \dots + (b_{-m} - a_{-m})t^{-(n+m)}] \cdot t^n > 0 \\
\iff & (b_n - a_n)t^n + (b_{n-1} - a_{n-1})t^{n-1} + \dots + (b_{-m} - a_{-m})t^{-m} > 0 \\
\iff & a_n t^n + a_{n-1}t^{n-1} + \dots + a_{-m}t^{-m} < b_n t^n + b_{n-1}t^{n-1} + \dots + b_{-m}t^{-m}.
\end{aligned}$$

□

By Lemma 2.3, a relation on $\mathbb{R}[t, t^{-1}]$ satisfying A1–A4 is determined uniquely as the lexicographic order. Such polynomials in $\mathbb{R}[t, t^{-1}]$ are called *lexicographically ordered polynomials.*

In this paper, we use the expression ω instead of t for an infinitely large number, and also ε instead of t^{-1} for an infinitely small number, so that

$$\text{(i) } \omega \cdot \varepsilon = 1 \quad \text{and} \quad \text{(ii) } \omega > k \text{ for all positive integer } k. \tag{10}$$

The ring of lexicographically ordered polynomials is, therefore, denoted by

$$\mathbb{R}[\omega, \varepsilon] = \{ a_n\omega^n + \dots + a_1\omega + a_0 + a_{-1}\varepsilon + \dots + a_{-m}\varepsilon^m \mid n, m \in \mathbb{N},\ a_n, \dots, a_{-m} \in \mathbb{R} \}, \tag{11}$$

where we tacitly assume that there is a relation $<$ on $\mathbb{R}[\omega, \varepsilon]$ satisfying A1–A4, identifying ω and ε with t and t^{-1}, respectively.

Also, in this paper, we use the following notations:

$$\begin{aligned}
\mathbb{R}[\varepsilon]_n &= \{ a_0 + a_1\varepsilon + \dots + a_n\varepsilon^n \mid a_0, a_1, \dots, a_n \in \mathbb{R} \}, \\
\mathbb{R}[\omega, \varepsilon]_n &= \{ a_n\omega^n + \dots + a_1\omega + a_0 + a_{-1}\varepsilon + \dots + a_{-n}\varepsilon^n \mid a_n, a_{n-1}, \dots, a_{-n} \in \mathbb{R} \}.
\end{aligned} \tag{12}$$

In other words, $\mathbb{R}[\omega, \varepsilon]_n$ is the set of lexicographically ordered polynomials whose degrees are not more than n and not less than $-n$.

Our notation $a_n\omega^n + \dots + a_1\omega + a_0 + a_{-1}\varepsilon + \dots + a_{-m}\varepsilon^m$ can be understood in a parallel way with the *decimal number system*. The decimal number system is indeed the lexicographical order. For example, in the decimal number system, the expression 24.13 is identified with the real number $2 \times 10 + 4 + 1 \times \frac{1}{10} + 3 \times (\frac{1}{10})^2$, and so on.

Remark 2.1 The readers who are familiar with "nonstandard analysis" will easily see that the above defined $\mathbb{R}[\omega, \varepsilon]$ can be identified with a proper subring of the hyperreal numbers. This is the reason why we use the symbols "ω" and "ε" for an infinitely large number and an infinitely small number, respectively, in our notation $a_n\omega^n + \dots + a_1\omega + a_0 + a_{-1}\varepsilon + \dots + a_{-m}\varepsilon^m$.

Formally, *the ordered field of hyperreal numbers* is constructed as an ultrapower of the real numbers, and hence it has such properties as (1) including $\mathbb{R}$ as a subfield, (2) containing an infinitely large number ω, (3) satisfying the transfer principle, i.e. satisfying the same first-order sentences as $\mathbb{R}$. For more details, see e.g. Goldblatt [3]. It is easy to verify that the lexicographically ordered polynomial ring $\mathbb{R}[\omega, \varepsilon]$ can be embedded into the ordered field of hyperreal numbers, identifying the variables ω and ε with an infinitely large number and its multiplicative inverse, respectively.

Hammond [4] introduced the smallest subfield containing both the real numbers IR and an infinitely small number ε, showing that this subfield is ordered lexicographically. Also Hammond [5] gave a suitable metric on this subfield so that it is extended to a complete metric space. □

3 Lexicographical Separation of Two Convex Cones

Let us recall that a subset C of $\mathbb{R}^n$ is *convex* if $\lambda x + (1-\lambda)y \in C$ whenever $x, y \in C$ and $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$. Let S be an arbitrary subset of $\mathbb{R}^n$ and let $c \in \mathbb{R}^n$. We say that *c is a linear convex combination of S* if, for some k, there are $\lambda_i > 0$ and $s_i \in S$ for $i = 1, \ldots, k$ with $\sum_{i=1}^{k} \lambda_i = 1$ such that $c = \sum_{i=1}^{k} \lambda_i s_i$.

Let P be a subset of $\mathbb{R}^n$. We say that P is a *convex cone* if for all $x, y \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$, (i) $x, y \in P$ implies $x + y \in P$, and (ii) $x \in P$ and $\lambda > 0$ imply $\lambda x \in P$. (It is easy to check that a convex cone P is a convex set.) A *positive cone P* is a convex cone in which $\mathbf{0}$ is not contained, i.e. (iii) $\mathbf{0} \notin P$ (where $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$). We denote by $\bar{P}$ the complement of P, that is, $\bar{P} = \mathbb{R}^n \setminus P$.

Ikeda [7] proved a lexicographical separation theorem that any two disjoint convex cones can be separated by linear functions and the lexicographic order. Below we present it without a proof.

Proposition 3.1 (Lexicographical Separation Theorem) Let P be a nonempty subset of $\mathbb{R}^n$. Suppose P is a positive cone and $\bar{P}$ is a convex cone. Then, there exist real-valued linear functions $g_1, \ldots, g_n$ on $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$,

$$x \in P \quad \text{iff} \quad (g_1(x), \ldots, g_n(x))^T >_L (0, \ldots, 0)^T. \tag{13}$$

□

Equivalent versions of this proposition were also proved by Hausner and Wendel [6], Klee [8], and Martínez-Legaz and Singer [11]; but the proofs were given by different methods. Proposition 3.1 will play a key role in establishing our main results in the next section.

Here we modify the statement of Proposition 3.1 for our later convenience:

Proposition 3.2 Let P be a nonempty subset of $\mathbb{R}^n$. Suppose P is a positive cone and $\bar{P}$ is a convex cone. Then there are lexicographically ordered polynomials $q_1, \ldots, q_n \in \mathbb{R}[\varepsilon]_{n-1}$ such that for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$,

$$(\lambda_1, \ldots, \lambda_n)^T \in P \quad \text{iff} \quad \lambda_1 q_1 + \cdots + \lambda_n q_n > 0. \tag{14}$$

□

Let us briefly check the equivalence of Proposition 3.1 and Proposition 3.2:

(Proposition 3.1 $\Rightarrow$ Proposition 3.2) Suppose there exist real-valued linear functions $g_1, \ldots, g_n$ on $\mathbb{R}^n$ such that (13) holds for all $x \in \mathbb{R}^n$. Let $(r_{ij})_{1 \le i,j \le n}$ be an $n \times n$ real matrix such that $r_{ij} = g_i(e_j)$ for all i, j, where e_j denotes the unit vector of $\mathbb{R}^n$ whose jth component is 1 while the others are 0. Let $q_1, \ldots, q_n \in \mathbb{R}[\varepsilon]_{n-1}$ be defined by

$$(q_1\ q_2\ \cdots\ q_n) = (1\ \varepsilon\ \ldots\ \varepsilon^{n-1}) \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}. \tag{15}$$

Then one can easily verifies that for all $\lambda_1,\dots,\lambda_n \in \mathrm{IR}$,

$$
\begin{aligned}
& (\lambda_1,\dots,\lambda_n)^T \in P \\
\Longleftrightarrow\quad & \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} >_L \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} && \text{(by (13))} \\
\Longleftrightarrow\quad & (1\ \varepsilon\ \dots\ \varepsilon^{n-1}) \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} > 0 && \text{(by Lemma 2.2)} \\
\Longleftrightarrow\quad & \lambda_1 q_1 + \cdots + \lambda_n q_n > 0. && \text{(by (15))}
\end{aligned}
$$

(Proposition 3.2 $\Rightarrow$ Proposition 3.1) Conversely, suppose there exist $q_1,\dots,q_n \in \mathrm{IR}[\varepsilon]_{n-1}$ such that (14) holds for all $\lambda_1,\dots,\lambda_n \in \mathrm{IR}$, where $q_1,\dots,q_n$ are represented by an $n\times n$ real matrix $(r_{ij})_{1\le i,j\le n}$ of the form (15). Let the corresponding linear functions $g_1,\dots,g_n$ on IR^n be defined as

$$g_j(\boldsymbol{x}) = \lambda_1 r_{j1} + \cdots + \lambda_n r_{jn} \qquad \text{for all } \boldsymbol{x} = (\lambda_1,\dots,\lambda_n)^T \in \mathrm{IR}^n. \tag{16}$$

Then, by a similar argument to the above, (13) holds for all $\boldsymbol{x} \in \mathrm{IR}$. □

4 Strict Linear Inequality Systems

In this section we apply the lexicographical separation theorem (Proposition 3.2) to strict linear inequality systems. First, we provide an extension of Gordan's lemma for a specific form of strict linear inequality systems, allowing an infinitely small number ε as a solution:

Theorem 4.1 *Let S be an arbitrary subset of IR^n. Then $\mathbf{0}$ is not a linear convex combination of S if and only if the system*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n > 0 \quad \text{for all } (a_1,a_2,\dots,a_n)^T \in S \tag{17}$$

has solutions $x_1,x_2,\cdots,x_n$ in $\mathrm{IR}[\varepsilon]_{n-1}$.

Proof. (the "if" part) Suppose the system (17) has solutions $x_1,x_2,\dots,x_n$ in $\mathrm{IR}[\varepsilon]_{n-1}$. Let $\boldsymbol{x} = (x_1,x_2,\dots,x_n)$. Suppose also $\mathbf{0}$ is a convex combination of S, i.e. there exist $\boldsymbol{s}_1,\dots,\boldsymbol{s}_k \in S$ and $\lambda_1,\dots,\lambda_k \in \mathrm{IR}$ with $\lambda_i > 0$ for $i = 1,\dots,k$ such that $\mathbf{0} = \lambda_1\boldsymbol{s}_1 + \cdots + \lambda_k\boldsymbol{s}_k$. Then

$$0 = \boldsymbol{x}\cdot\mathbf{0} = \boldsymbol{x}\cdot\Big(\sum_{i=1}^{k}\lambda_i\boldsymbol{s}_i\Big) = \sum_{i=1}^{k}\lambda_i(\boldsymbol{x}\cdot\boldsymbol{s}_i) > 0,$$

a contradiction.

(the "only if" part) Let S be a subset of IR^n, and suppose $\mathbf{0}$ is not a convex combination of S. We shall construct a subset P of IR^n in such a way that $P \supseteq S$, P is a positive cone, and $\bar{P}$ is a convex cone in IR^n. Once the existence of such P is confirmed, by Proposition 3.2 there exist polynomials $q_1,\dots,q_n \in \mathrm{IR}[\varepsilon]_{n-1}$ such that (14) holds for all $(\lambda_1,\dots,\lambda_n)^T \in \mathrm{IR}^n$, and hence $(x_1,\dots,x_n) = (q_1,\dots,q_n)$ is a desired solution to (17).

By Zorn's lemma, there exists a maximal subset P of IR^n (with respect to inclusion) such that

$$\text{(i)}\quad P \supseteq S, \qquad \text{(ii)}\quad \mathbf{0} \text{ is not a convex combination of } P. \tag{18}$$

Then, the following conditions hold for P:

(1) $\quad \mathbf{0} \notin P$,

(2) $\quad$ for all $\boldsymbol{x} \in \mathbb{R}^n$, $\quad \boldsymbol{x} \in P$ or $\boldsymbol{x} = \mathbf{0}$ or $-\boldsymbol{x} \in P$,

(3) $\quad$ for all $\boldsymbol{x} \in \mathbb{R}^n$, $\quad \boldsymbol{x} \in P$ and $\lambda > 0$ imply $\lambda \boldsymbol{x} \in P$,

(4) $\quad$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\quad \boldsymbol{x} \in P$ and $\boldsymbol{y} \in P$ imply $\boldsymbol{x} + \boldsymbol{y} \in P$.

(1) is trivial. We shall verify (2), (3), and (4) as follows.

(2): Let $\boldsymbol{x}$ be a nonzero vector in $\mathbb{R}^n$. We shall show that either $\boldsymbol{x}$ or $-\boldsymbol{x}$ is in P. Suppose otherwise: $\boldsymbol{x} \notin P$ and $-\boldsymbol{x} \notin P$. Then, by the maximality of P with respect to (18), $\mathbf{0}$ is a convex combination of $P \cup \{\boldsymbol{x}\}$, and also $\mathbf{0}$ is a convex combination of $P \cup \{-\boldsymbol{x}\}$. Hence, there exist $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_l, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_m \in P$ and $\lambda_0, \ldots, \lambda_l, \mu_0, \ldots, \mu_m \in \mathbb{R}$ such that

$$\lambda_0 \boldsymbol{x} + \sum_{k=1}^{l} \lambda_k \boldsymbol{p}_k = \mathbf{0}, \lambda_i > 0 \quad \text{for } i = 0, \ldots, l, \quad \text{and} \quad \sum_{i=0}^{l} \lambda_i = 1,$$

$$\mu_0(-\boldsymbol{x}) + \sum_{k=1}^{m} \mu_k \boldsymbol{q}_k = \mathbf{0}, \mu_j > 0 \quad \text{for } j = 0, \ldots, m, \quad \text{and} \quad \sum_{i=0}^{m} \mu_i = 1.$$

Therefore $\mu_0(\sum_{k=1}^{l} \lambda_k \boldsymbol{p}_k) + \lambda_0(\sum_{k=1}^{m} \mu_k \boldsymbol{q}_k) = \mathbf{0}$. Let $\xi = \mu_0(\sum_{k=1}^{l} \lambda_k) + \lambda_0(\sum_{k=1}^{m} \mu_k)$. Then

$$\frac{1}{\xi} \mu_0 \left(\sum_{k=1}^{l} \lambda_k \boldsymbol{p}_k \right) + \frac{1}{\xi} \lambda_0 \left(\sum_{k=1}^{m} \mu_k \boldsymbol{q}_k \right) = \mathbf{0},$$

which means that $\mathbf{0}$ is a convex combination of P, a contradiction.

(3): Suppose $\boldsymbol{x} \in P$, $\lambda > 0$, and $\lambda \boldsymbol{x} \notin P$. Then, by the maximality of P with respect to (18), $\mathbf{0}$ is a convex combination of $P \cup \{\lambda \boldsymbol{x}\}$. Hence, there exist $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_l \in P$ and $\mu_0, \ldots, \mu_l \in \mathbb{R}$ such that

$$\mu_0(\lambda \boldsymbol{x}) + \sum_{k=1}^{l} \mu_k \boldsymbol{p}_k = \mathbf{0}, \qquad \mu_i > 0 \quad \text{for } i = 0, \ldots, l, \quad \text{and} \quad \sum_{i=0}^{l} \mu_i = 1.$$

Let $\xi = \mu_0 \lambda + \sum_{k=1}^{l} \mu_k$. Then $\frac{1}{\xi}(\mu_0 \lambda \boldsymbol{x} + \sum_{k=1}^{l} \mu_k \boldsymbol{p}_k) = \mathbf{0}$. This means that $\mathbf{0}$ is a convex combination of P, a contradiction.

(4): Suppose there exist $\boldsymbol{x}$, $\boldsymbol{y}$ such that $\boldsymbol{x} \in P$, $\boldsymbol{y} \in P$, and $\boldsymbol{x} + \boldsymbol{y} \notin P$. By (2), we have either $\boldsymbol{x} + \boldsymbol{y} = \mathbf{0}$ or $-\boldsymbol{x} - \boldsymbol{y} \in P$; both cases contradict the hypothesis that $\mathbf{0}$ is not a convex combination of P.

Thus, by (1), (3), and (4), P is a positive cone in $\mathbb{R}^n$. Now, by (1) and (2), we have exactly one of $\boldsymbol{x} \in P$, $\boldsymbol{x} = \mathbf{0}$, and $\boldsymbol{x} \in -P$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Hence $\bar{P} = -P \cup \{\mathbf{0}\}$. Therefore $\bar{P}$ is a convex cone in $\mathbb{R}^n$. □

Example 4.1 Let S_1 ($\subseteq \mathbb{R}^3$) be

$$S_1 = \{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\} \cup \{(1,-1,0)^T, (1,-2,0)^T, \ldots, (1,-k,0)^T, \ldots\} \cup \{(0,1,-1)^T, (0,1,-2)^T, \ldots, (0,1,-k)^T, \ldots\}. \tag{19}$$

It is easy to verify that $(0,0,0)^T$ is not a linear convex combination of S_1. By Theorem 4.1, the corresponding system

$$x > 0,\ y > 0,\ z > 0, \quad x - y > 0,\ x - 2y > 0,\ \ldots,\ x - ky > 0,\ \ldots, \quad y - z > 0,\ y - 2z > 0,\ \ldots,\ y - kz > 0,\ \ldots \tag{20}$$

has solutions x, y, z in $\mathbb{R}[\varepsilon]_2$: for example, $(x, y, z) = (1, \varepsilon, \varepsilon^2)$. □

Next we deal with the general form of strict linear inequality systems :

Theorem 4.2 (Main Theorem) *Let S be an arbitrary subset of $\mathbb{R}^{n+1}$. Then $\mathbf{0}$ is not a linear convex combination of $S \cup \{(0,\ldots,0,-1)^T\}$ if and only if the system*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n > b \quad \text{for all } (a_1,a_2,\ldots,a_n,b)^T \in S \tag{21}$$

has solutions $x_1,x_2,\cdots,x_n$ in $\mathbb{R}[\omega,\varepsilon]_n$.

The proof of Theorem 4.2 will be given by a series of lemmas. Let us denote by $\hat{S}$ the subset of $\mathbb{R}^{n+1}$ whose elements are obtained from S by changing the signs of the last components, i.e.

$$\hat{S} = \{(a_1,\ldots,a_n,-b)^T \in \mathbb{R}^{n+1} \mid (a_1,\ldots,a_n,b)^T \in S\}. \tag{22}$$

Similarly, for $\boldsymbol{a} = (a_1,\ldots,a_n,b)^T$ we use the notation $\hat{\boldsymbol{a}} = (a_1,\ldots,a_n,-b)^T$.

Lemma 4.1 Let S be a subset of $\mathbb{R}^{n+1}$. Then $\mathbf{0}$ is a linear convex combination of S if and only if $\mathbf{0}$ is a linear convex combination of $\hat{S}$.

Proof. Suppose $\mathbf{0}$ is a linear convex combination of S. That is, for some k, there are $\lambda_i > 0$ and $\boldsymbol{s}_i \in S$ for $i = 1,\ldots,k$ with $\sum_{i=1}^k \lambda_i = 1$ such that $\mathbf{0} = \sum_{i=1}^k \lambda_i \boldsymbol{s}_i$. Then, one can easily verifies that $\mathbf{0} = \sum_{i=1}^k \lambda_i \hat{\boldsymbol{s}}_i$, which means that $\mathbf{0}$ is a linear convex combination of $\hat{S}$. The converse can also be shown in a similar way. □

Lemma 4.2 Let S be a subset of $\mathbb{R}^{n+1}$. Suppose $\mathbf{0}$ is not a linear convex combination of $S \cup \{(0,\ldots,0,1)^T\}$. Then, the system

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + by > 0 \quad \text{for all } (a_1,a_2,\ldots,a_n,b)^T \in S \tag{23}$$

has solutions $x_1,x_2,\cdots,x_n,y$ in $\mathbb{R}[\varepsilon]_n$ such that $y = c\varepsilon^k$ for some real number $c > 0$ and some $k \in \mathbb{N}$ with $k \le n$.

Proof. Let S be a subset of $\mathbb{R}^{n+1}$. Suppose $\mathbf{0}$ is not a linear convex combination of $S \cup \{(0,\ldots,0,1)^T\}$. Then, by Theorem 4.1, the system

$$y > 0 \quad \text{and} \tag{24}$$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + by > 0 \quad \text{for all } (a_1,a_2,\ldots,a_n,b)^T \in S \tag{25}$$

has solutions $x'_1,x'_2,\ldots,x'_n,y'$ in $\mathbb{R}[\varepsilon]_n$. Note that $x'_1,x'_2,\ldots,x'_n,y'$ can be expressed as

$$(x'_1\ x'_2\ \ldots\ x'_n\ y') = (1\ \varepsilon\ \ldots\ \varepsilon^n)\begin{pmatrix} x'_{11} & \cdots & x'_{1n} & y'_1 \\ \vdots & \ddots & \vdots & \vdots \\ x'_{(n+1)1} & \cdots & x'_{(n+1)n} & y'_{n+1} \end{pmatrix} \tag{26}$$

for some real numbers x'_{ij} and y'_i. Let $\boldsymbol{x}'_j = (x'_{1j},\ldots,x'_{(n+1)j})^T$ for $j = 1,\ldots,n$ and $\boldsymbol{y}' = (y'_1,\ldots,y'_{n+1})^T$. Then, the system (25) can be rewritten as

$$(x'_1\ \ldots\ x'_n\ y')(a_1,\ldots,a_n,b)^T = (1\ \varepsilon\ \ldots\ \varepsilon^n)(\boldsymbol{x}'_1\ \ldots\ \boldsymbol{x}'_n\ \boldsymbol{y}')(a_1,\ldots,a_n,b)^T > 0$$

for all $(a_1,\ldots,a_n,b)^T \in S$. By Lemma 2.2, this means that

$$(\boldsymbol{x}'_1\ \ldots\ \boldsymbol{x}'_n\ \boldsymbol{y}')(a_1,\ldots,a_n,b)^T >_L \mathbf{0} \quad \text{for all } (a_1,\ldots,a_n,b)^T \in S. \tag{27}$$

Also, by (24), we have $\boldsymbol{y}' = (y'_1, \ldots, y'_{n+1})^T >_L \mathbf{0}$. Hence there is a positive integer d with $d \leq n+1$ such that

$$y'_1 = \cdots = y'_{d-1} = 0 \quad \text{and} \quad y'_d > 0. \tag{28}$$

Let us define an $(n+1) \times (n+1)$ real matrix $C = (c_{ij})$ by

$$c_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ -y'_i / y'_d & \text{if} \quad j = d \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

It is easy to check that C is a unitary lower triangular matrix. Hence, by Proposition 2.2, the system (27) implies that

$$C(\boldsymbol{x}'_1 \ \ldots \ \boldsymbol{x}'_n \ \boldsymbol{y}')(a_1, \ldots, a_n, b)^T >_L \mathbf{0} \quad \text{for all } (a_1, \ldots, a_n, b)^T \in S. \tag{30}$$

Let $\boldsymbol{x}_j = C\boldsymbol{x}'_j$ for $j = 1, \ldots, n$, and $\boldsymbol{y} = C\boldsymbol{y}'$. Define $x_1, \ldots, x_n, y \in \mathbb{R}[\varepsilon]_n$ by

$$\begin{aligned} x_j &= (1 \ \varepsilon \ \ldots \ \varepsilon^n)\boldsymbol{x}_j \quad \text{for } j = 1, \ldots, n, \\ y &= (1 \ \varepsilon \ \ldots \ \varepsilon^n)\boldsymbol{y}. \end{aligned} \tag{31}$$

One can easily verifies that, by (30) and by Lemma 2.2, these $x_1, \ldots, x_n, y$ give solutions to (23). It remains to show that $y = c\varepsilon^k$ for some $c > 0$ and some $k \leq n$.

Let $\boldsymbol{y} = (y_1, \ldots, y_{n+1})^T$. We shall show that $y_d = y'_d (> 0)$ and $y_i = 0$ for all $i \neq d$. This means $y = y_d \varepsilon^{d-1}$ by (31).

By (29) we obtain $y_d = \sum_{j=1}^{n+1} c_{dj} y'_j = c_{dd} y'_d = y'_d$. Also, for all $i > d$,

$$y_i = \sum_{j=1}^{n+1} c_{ij} y'_j = c_{ii} y'_i + \sum_{j<i} c_{ij} y'_j = y'_i + c_{id} y'_d = y'_i + \left(-\frac{y'_i}{y'_d}\right) y'_d = 0.$$

For all $i < d$,

$$y_i = \sum_{j=1}^{n+1} c_{ij} y'_j = c_{ii} y'_i + \sum_{j<i} c_{ij} y'_j = y'_i = 0 \ \text{(by (28))}.$$

Thus we get the conclusion. □

Proof of Theorem 4.2. Let $\boldsymbol{\delta}_{n+1} = (0, \ldots, 0, 1)^T \in \mathbb{R}^{n+1}$. It is enough to show, by Lemma 4.1, that $\mathbf{0}$ is not a linear convex combination of $\hat{S} \cup \{\boldsymbol{\delta}_{n+1}\}$ if and only if the system

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b > 0 \quad \text{for all } (a_1, a_2, \ldots, a_n, b)^T \in \hat{S} \tag{32}$$

has solutions $x_1, x_2, \cdots, x_n$ in $\mathbb{R}[\omega, \varepsilon]_n$.

For the "if" part, suppose the system (32) has solutions $x_1, x_2, \cdots, x_n$ in $\mathbb{R}[\omega, \varepsilon]_n$. Suppose also $\mathbf{0}$ is a linear convex combination of $\hat{S} \cup \{\boldsymbol{\delta}_{n+1}\}$, i.e. there exist $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k \in \hat{S}$ and $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$ with $\lambda_i \geq 0$ for $i = 0, 1, \ldots, k$ and $\sum_{i=0}^k \lambda_i = 1$ such that $\mathbf{0} = \lambda_0 \boldsymbol{\delta}_{n+1} + \lambda_1 \boldsymbol{s}_1 + \cdots + \lambda_k \boldsymbol{s}_k$. Note that at least one of $\lambda_0, \lambda_1, \ldots, \lambda_k$ is strictly positive. Let $\boldsymbol{x} = (x_1, \ldots, x_n, 1)$. Then,

$$0 = \boldsymbol{x} \cdot \mathbf{0} = \boldsymbol{x} \cdot (\lambda_0 \boldsymbol{\delta}_{n+1} + \sum_{i=1}^k \lambda_i \boldsymbol{s}_i) = \lambda_0 + \sum_{i=1}^k \lambda_i (\boldsymbol{x} \cdot \boldsymbol{s}_i) > 0,$$

a contradiction.

For the "only if" part, suppose $\mathbf{0}$ is not a linear convex combination of $\hat{S} \cup \{\delta_{n+1}\}$. By Lemma 4.2, the system

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + by > 0 \quad \text{for all } (a_1, a_2, \ldots, a_n, b)^T \in \hat{S} \tag{33}$$

has solutions $x_1, x_2, \cdots, x_n, y$ in $\mathbb{R}[\varepsilon]_n$ such that $y = c\varepsilon^k$ for some real number $c > 0$ and some $k \in \mathbb{N}$ with $k \leq n$. Hence, by multiplying $\omega^k/c\ (> 0)$ to the inequality of (33), we obtain

$$a_1\left(\frac{x_1\omega^k}{c}\right) + \cdots + a_n\left(\frac{x_n\omega^k}{c}\right) + b > 0 \quad \text{for all } (a_1, \ldots, a_n, b)^T \in \hat{S}. \tag{34}$$

Since $x_1, x_2, \cdots, x_n \in \mathbb{R}[\varepsilon]_n$ we have $x_1\omega^k/c, \ldots, x_n\omega^k/c \in \mathbb{R}[\omega, \varepsilon]_n$. This means that the system (32) gets required solutions. □

Example 4.2 Let $S_2\ (\subseteq \mathbb{R}^3)$ be

$$S_2 = \{(0,1,k)^T \mid k \in \mathbb{N}\} \cup \{(1,-1,0)^T\} \cup \{(-1,1,-\frac{1}{k})^T \mid k \in \mathbb{N}_+\}. \tag{35}$$

The corresponding system of strict linear inequalities can be expressed as

$$y > k \text{ for all } k \in \mathbb{N}, \quad x - y > 0, \text{ and } x - y < \frac{1}{k} \text{ for all positive integer } k. \tag{36}$$

It is not difficult to verify that $(0,0,0)^T$ is not a linear convex combination of $S_2 \cup \{(0,0,-1)^T\}$. Hence, by Theorem 4.2, the above system (36) has solutions x, y in $\mathbb{R}[\omega, \varepsilon]_2$. For example, $(x, y) = (\omega + \varepsilon, \omega)$. □

5 Concluding Remarks

The main result of this paper is Theorem 4.2, giving a necessary and sufficient condition for the existence of solutions to infinite systems of strict linear inequalities.

Here we adopt $\mathbb{R}[\omega, \varepsilon]_n$ as the domain of solutions. But, as one can easily notice through the careful reading of the proof of Theorem 4.2, the domain of solutions can be restricted to $\{a_k\omega^k + \cdots + a_1\omega + a_0 + a_{-1}\varepsilon + \cdots + a_{-m}\varepsilon^m \mid a_k, \ldots, a_{-m} \in \mathbb{R}\}$ for some $k, m \in \mathbb{N}$ with $k, m \leq n$ and $k + m \leq n$. It is also easy to check that Theorem 4.2 remains valid whenever the domain of solutions is an ordered vector space over $\mathbb{R}$ containing $\mathbb{R}[\omega, \varepsilon]_n$ as a subspace.

Theorem 4.2 is considered as an extension of Gordan's lemma, and is proved by way of the lexicographical separation theorem (Proposition 3.1). It will be interesting to deduce also a Farkas type theorem from the lexicographical separation theorem, and to apply it to non-linear programming theory.

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An Extension of Gordan's Lemma for Infinite Systems of Strict Linear Inequalities

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It is well-known that Gordan's lemma, one of the theorems of the alternatives, gives a necessary and sufficient condition for the existence of solutions to finite systems of strict linear inequalities. The aim of this paper is to show that Gordan's lemma can be extended for the infinite case if we adopt a suitable non-Archimedean structure as the domain of solutions to infinite systems of strict linear inequalities. For this purpose we introduce an extended structure of the field of real numbers $\mathbb{R}$ with both an infinitely large number ω and an infinitely small number ε, showing that this structure coincides with the set of lexicographically ordered vectors. The main result is derived as a consequence of the lexicographical separation theorem of Hausner and Wendel, Klee, Martínez-Legaz and Singer that any two disjoint convex sets in $\mathbb{R}^n$ can be separated lexicographically.

Key Words : linear inequalty systems, lexicographic orders, non-Archimedean structures