An Approach to Housing Markets with Externalities using Tarski's Fixed-Point Theorem

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1 Introduction

Several authors including Kaneko [2], Kaneko and Yamamoto [3] investigated mathematical models of rental housing markets as applications of the assignment game developed by Shapley and Shubik [4]. In their models, housing markets are formulated as exchange economies in which households and landlords trade an apartment as an indivisible commodity for a perfectly divisible commodity called money. Kaneko [2] gave a recursive equation that determines a competitive equilibrium in a housing market model; Kaneko and Yamamoto [3] proved the existence of competitive equilibrium in a more generalized model of an assignment market. These models, however, are not enough to capture direct interactions among households, which should be treated in the form of externalities.

On this account, Kaneko remarked that a competitive equilibrium exists even when a special type of externalities is introduced into the housing market model (Section 5 in [2]). In his formulation, the externalities among households are considered in such a way that the average income of households has an effect on a household's consumption behavior. He formulated such a model of the housing market with externalities as a straightforward extension of that without externalities. Nevertheless, his assumptions includes a rather technical condition (Assumption F' in [2]) saying that the apartments are graded according to the qualities under a special allocation of households.

In this paper, we propose a model of rental housing markets with externalities as a modification of that of Kaneko [2]. We modify the assumptions of Kaneko by discarding the above condition (Assumption F' in [2]) and introducing much simpler conditions, namely, (i) a high average income is preferred to a low one, and (ii) there are two housing areas in the market. Under our modified assumptions we give a recursive equation that determines a competitive equilibrium in the housing market with externalities. We also show that in our model the derivation of the recursive equation can be achieved by the use of Tarski's fixed-point theorem.

This paper is organized as follows. In Section 2 we give a model of a rental housing market with externalities, assuming that there are only two housing areas. Also we give a recursive equation that determines a competitive equilibrium. Our derivation of the recursive equation depends on a lemma saying that there exists an allocation of households under which the apartments are graded according to the qualities. The proof of this existence lemma is postponed until the next section. In Section 3 we review Tarski's fixed-point theorem, and provide the proof of the existence lemma in Section 2.

2 A Rental Housing Market on Two Areas

Let us consider a *rental housing market model* as an exchange economy (M,N), where $M = \{1,2,\dots,m\}$ is the set of all landlords and $N = \{1',2',\dots,n'\}$ is the set of all households. In this economy each landlord supply some units of apartments on the market, while each household looks for one unit of apartment. The apartments are classified into s types, $1,\dots,s$. For each $h = 1,\dots,s$, the apartment of type h is represented by the s-dimensional unit vector e^h with $e^h_j = 1$ if $j = h$ and $e^h_j = 0$ if $j \neq h$. Every landlord $i \in M$ is endowed with $w^i = (w^i_1,\dots,w^i_s)$ units of apartments that he can lease to households, where w^i is a nonnegative integer for all $i \in M$. We assume

$$\text{(A)} \qquad \sum_{i\in M} w^i_k > 0 \text{ for all } k = 1,\dots,s \quad \text{and} \quad \sum_{k=1}^{s}\sum_{i\in M} w^i_k \geq n.$$

Every household $j \in N$ is endowed with I^j (>0) amount of money but with no unit of apartment. We order the households as

$$I_1 \geq I_2 \geq \cdots \geq I_n. \tag{1}$$

A simple utility function is assumed for landlords. Each landlord $i \in M$ has the *evaluation function* $u^i(x)$ on the set $X^i \equiv \{x = (x_1,\dots,x_s) \in I\!N^s \mid 0 \leq x_k \leq w^i_k \text{ for all } k = 1,\dots,s\}$ such that for all $x \in X^i$,

$$\text{(B)} \qquad u^i(x) = \sum_{k=1}^{s} u^i(x_k e^k) \quad \text{and} \quad u^i(x_k e^k) = a_k x_k \text{ for all } k = 1,\dots,s,$$

where a_k is a positive number. We assume that a_k is independent of landlords.

Hence, if landlord i leases $w^i - x$ units of apartments at rents $r_1, r_2, \dots, r_s$, his utility (profit) is

$$u^i(x) + \sum_{k=1}^{s} r_k(w^i_k - x_k) = \sum_{k=1}^{s} a_k x_k + \sum_{k=1}^{s} r_k(w^i_k - x_k). \tag{2}$$

We call a_k the *evaluation value of the kth apartment.*

Let τ be a partition of $\{e^1,e^2,\dots,e^s\}$. Each $S \in \tau$ means a housing area in which apartments in S are located. For each type of apartments $h = 1,\dots,s$, we denote by $S(h)$ the area $S \in \tau$ such that $e^h \in S$. We assume that every household's preference depends on his apartment, consumption, and the environment of $S \in \tau$ in which his apartment is located. The environment of S can be represented by the average income of the households who rent apartments in S. More precisely, let $x = (x^{1'},\dots,x^{n'}) \in \{0,e^1,e^2,\dots,e^s\}^n$ be an allocation of households satisfying $\sum_{j\in N} x^j_k \leq \sum_{i\in M} w^i_k$ for all $k = 1,\dots,s$. For each type of apartments $h = 1,\dots,s$, we define *the average income $\gamma_h(x)$ in the area $S(h)$ under the allocation x* by

$$\gamma_h(x) = \begin{cases} \dfrac{\sum_{x^j\in S(h)} I_j}{\sum_{i\in M}\sum_{e^k\in S(h)} w^i_k} & \text{if } \sum_{i\in M}\sum_{e^k\in S(h)} w^i_k \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We assume that, for each type of apartments $h = 1,\dots,s$, the environment of $S(h)$ is represented by the average income $\gamma_h(x)$.

Each household $j \in N$ has the same preference relation $\succeq$ on $Y \equiv \{0, e^1, e^2, \dots, e^s\} \times \mathrm{IR}_+ \times \mathrm{IR}_+$, where IR_+ is the set of nonnegative real numbers. For convenience, we may use $e^{s+1} = 0$, $a_{s+1} = 0$, and $\gamma_{s+1}(x) = 0$ for any x. The expression $(e^h, m, \eta) \in Y$ means that the household j rents one unit of the apartment of type h and enjoys the consumption m of money in his environment η. We assume that

(C) $\succeq$ is a complete preorder on Y.

The strict preference $\succ$ and the indifference relation $\sim$ are defined in the usual ways, i.e. $a \succ b$ iff not $(b \succeq a)$; $a \sim b$ iff $a \succeq b$ and $b \succeq a$. We make the following assumptions on the households' preference relation $\succeq$:

(D) For all $(x,m,\eta) \in Y$, if $\delta > 0$ then $(x, m+\delta, \eta) \succ (x,m,\eta)$.
(E) For all $(x,m,\eta) \in Y$, if $\delta \geq 0$ then $(x, m, \eta+\delta) \succeq (x,m,\eta)$.
(F) If $(x,m_1,\eta_1) \succ (y,m_2,\eta_2)$, then there is an $m_3 \geq 0$ such that $(x,m_1,\eta_1) \sim (y,m_3,\eta_2)$.
(G) If $(x,m_1,\eta_1) \sim (y,m_2,\eta_2)$, $m_1 < m_2$ and $\delta > 0$, then $(x, m_1+\delta, \eta_1) \succ (y, m_2+\delta, \eta_2)$.

The assumptions (D) and (F) need no explanation. The assumption (E) means that a high average income is preferred to a low one. The assumption (G) means that the marginal utility of money is diminishing.

Lemma 2.1 (i) *If $(x,m_1,\eta_1) \sim (y,m_2,\eta_2)$ and $0 < \delta \leq m_2 < m_1$, then $(x, m_1-\delta, \eta_1) \succ (y, m_2-\delta, \eta_2)$.* (ii) *If $(x,m_1,\eta_1) \succeq (y,m_1,\eta_2)$ and $(x,m_1,\eta_1) \sim (y,m_2,\eta_2)$, then $m_1 \leq m_2$.* (iii) *If $(x,m_1,\eta_1) \succ (y,m_1,\eta_2)$, then $(x,m_2,\eta_1) \succ (y,m_2,\eta_2)$ for all $m_2 > m_1$.*

Proof. See Appendix. □

Let π be a permutation on $\{1,2,\dots,s\}$. We define a function $G_\pi(k)$ $(k = 1,\dots,s)$ by

$$G_\pi(k) = \sum_{t=1}^{k} \sum_{i \in M} w^i_{\pi(t)}. \tag{4}$$

For convenience, we put $G_\pi(0) = 0$. Let $v = (v^{1'}, \dots, v^{n'}) \in \{0, e^1, e^2, \dots, e^s\}^n$ be defined by

$$v^j = e^{\pi(k)} \quad \text{if } G_\pi(k-1) < j \leq G_\pi(k). \tag{5}$$

Note that, by the assumption (A), for each $j = 1,\dots,n$ there is a k such that $G_\pi(k-1) < j \leq G_\pi(k)$.

The following lemma states that, if there are only two housing areas in the market, then there exists an allocation of households under which the apartments are graded according to the qualities. This lemma plays a crucial role in the following argument.

Lemma 2.2 (Main Lemma) *Suppose there are two housing areas in the market. Then, there exists a permutation π on $\{1,2,\dots,s\}$ such that*

$$(e^{\pi(1)}, m, \gamma_{\pi(1)}(v)) \succeq (e^{\pi(2)}, m, \gamma_{\pi(2)}(v)) \succeq \cdots \succeq (e^{\pi(s)}, m, \gamma_{\pi(s)}(v)) \quad \text{for all } m \geq 0, \tag{6}$$

where v is the allocation of households defined by (5). □

We will give a proof of this lemma using Tarski's fixed-point theorem in Section 3.

In the following, we assume that

(H) there are two housing areas in the market, i.e. τ is a separation (S_I, S_{II}) of $\{e^1, e^2, \ldots, e^s\}$ such that $S_I, S_{II} \subseteq \{e^1, e^2, \ldots, e^s\}$, $S_I \cap S_{II} = \emptyset$, and $S_I \cup S_{II} = \{e^1, e^2, \ldots, e^s\}$.

By renaming types of apartments suitably, we shall assume without loss of generality that the identity permutation π_{id} satisfies (6) of Lemma 2.2. Thus

$$(e^1, m, \gamma_1(v)) \succeq (e^2, m, \gamma_2(v)) \succeq \cdots \succeq (e^s, m, \gamma_s(v)) \quad \text{for all } m \geq 0, \tag{7}$$

where $v = (v^{1'}, \ldots, v^{n'})$ is the allocation of households defined by

$$v^j = e^k \quad \text{if } G_{\pi_{\text{id}}}(k-1) < j \leq G_{\pi_{\text{id}}}(k). \tag{8}$$

We shall denote the function $G_{\pi_{\text{id}}}(k)$ by $G(k)$. From the assumption (A) we have

$$G(f-1) < n \leq G(f) \quad \text{for some } f \leq s. \tag{9}$$

Now we define a rent vector $(p_1, \ldots, p_{f-1})$ backward recursively by

$$\begin{aligned} (e^f, I_{G(f-1)} - a_f, \gamma_f(v)) &\sim (e^{f-1}, I_{G(f-1)} - p_{f-1}, \gamma_{f-1}(v)) \\ (e^{f-1}, I_{G(f-2)} - p_{f-1}, \gamma_{f-1}(v)) &\sim (e^{f-2}, I_{G(f-2)} - p_{f-2}, \gamma_{f-2}(v)) \\ &\vdots \\ (e^2, I_{G(1)} - p_2, \gamma_2(v)) &\sim (e^1, I_{G(1)} - p_1, \gamma_1(v)). \end{aligned} \tag{10}$$

We shall show that this rent vector $(p_1, \ldots, p_{f-1})$ forms a competitive rent vector under appropriate assumptions. The following lemma provides a condition for (10) to have a unique solution.

Lemma 2.3 *If $I_n \geq a_f$ and $(e^f, I_n - a_f, \gamma_f(v)) \succeq (e^1, 0, \gamma_1(v))$, then there is a unique vector $(p_1, \ldots, p_{f-1})$ that satisfies (10) and has the property*

$$p_1 \geq p_2 \geq \cdots \geq p_{f-1} \geq a_f. \tag{11}$$

Proof. This is proved in a similar way to Lemma 2 of Kaneko [2]. See Appendix. □

We assume the following conditions:

(I) $I_n \geq a_f$, $(e^f, I_n - a_f, \gamma_f(v)) \succ (e^1, 0, \gamma_1(v))$, and $(e^f, I_n - a_f, \gamma_f(v)) \succ (e^k, I_n - a_k, \gamma_k(v))$ for all k $(f < k \leq s+1)$ with $I_n \geq a_k$.

(J) $p_k > a_k$ for all $k = 1, \ldots, f-1$.

Lemma 2.4 *Let I be an income level such that $I_{G(k-1)+1} \geq I \geq I_{G(k)}$ $(1 \leq k \leq f-1)$. Then the following propositions hold:*

(i) $(e^k, I - p_k, \gamma_k(v)) \succeq (e^{k+1}, I - p_{k+1}, \gamma_{k+1}(v)) \succeq \cdots \succeq (e^{f-1}, I - p_{f-1}, \gamma_{f-1}(v)) \succeq (e^f, I - a_f, \gamma_f(v))$ *and* $(e^f, I - a_f, \gamma_f(v)) \succ (e^t, I - a_t, \gamma_t(v))$ *for all $t > f$ with $I \geq a_t$.*

(ii) $(e^k, I - p_k, \gamma_k(v)) \succeq (e^{k-1}, I - p_{k-1}, \gamma_{k-1}(v)) \succeq \cdots \succeq (e^{t+1}, I - p_{t+1}, \gamma_{t+1}(v)) \succeq (e^t, I - p_t, \gamma_t(v))$ *if $p_t \leq I$.*

Proof. These are proved in similar ways to those in Lemma 3 of Kaneko [2]. See Appendix. □

We say that $(r,x,y,\eta) = (r_1,\dots,r_s,x^1,\dots,x^m,y^{1'},\dots,y^{n'},\eta_1,\dots,\eta_s)$ is a *competitive equilibrium* if

(a) $r \in \mathbb{R}^s_+$, $\eta \in \mathbb{R}^s_+$, $x^i \in X^i$ for all $i \in M$, and $y^j \in \{0,e^1,e^2,\dots,e^s\}$ for all $j \in N$;

(b) for all $i \in M$

$$u^i(x^i) + r(w^i - x^i) \;=\; \max_{z \in X^i}(u^i(z) + r(w^i - z));$$

(c) for all $j \in N$ and all $z \in \{0,e^1,e^2,\dots,e^s\}$ with $rz \leq I^j$,

$$(y^j, I^j - ry^j, \eta y^j) \succeq (z, I^j - rz, \eta z);$$

(d) $\sum_{i \in M} x^i + \sum_{j \in N} y^j \;=\; \sum_{i \in M} w^i$;

(e) $\eta \;=\; (\gamma_1(y),\dots,\gamma_s(y))$.

We call $r = (r_1,\dots,r_s)$ a *competitive rent vector* and each r_k $(k = 1,\dots,s)$ a *competitive rent* of the kth apartment.

The condition (a) means that each variable belongs to the appropriate set. The condition (b) is the utility (profit) maximization of the landlords. The condition (c) is the utility maximization of the households under the budget constraints. The condition (d) is the equivalence of total supplies and total demands of apartments. The condition (e) means that the vector of externalities $\eta = (\eta_1,\dots,\eta_s)$ corresponds to the vector of the average incomes.

The following theorem ensures the existence of a competitive equilibrium in our housing market model.

Theorem 2.1 *Assume (A) – (H). Let* $r = (r_1,\dots,r_s)$ *be defined by*

$$\begin{aligned} r_k &= p_k && \text{if } k < f \\ &= a_k && \text{otherwise,} \end{aligned} \tag{12}$$

where $p_1,\dots,p_{f-1}$ *are chosen by (10). Then* $r = (r_1,\dots,r_s)$ *is a competitive rent vector.* □

Proof. Since Lemma 2.3 ensures the existence and the uniqueness of $(p_1,\dots,p_{f-1})$, it is sufficient to construct a vector $(x,y,\eta) = (x^1,\dots,x^m,y^{1'},\dots,y^{n'},\eta_1,\dots,\eta_s)$ and to show that (r,x,y,η) is a competitive equilibrium.

We define $(x,y,\eta) = (x^1,\dots,x^m,y^{1'},\dots,y^{n'},\eta_1,\dots,\eta_s)$ by

$$y^j \;=\; v^j \quad \text{for } j = 1,\dots,n;$$

$$\eta_h \;=\; \gamma_h(v) \quad \text{for } h = 1,\dots,s;$$

$$x^i_k \;=\; \begin{cases} 0 & \text{if } k < f, \\ w^i_k & \text{if } k > f, \end{cases} \qquad \begin{array}{l} \text{and } x^i_f \text{ is a positive integer with } 0 \leq x^i_f \leq w^i_f \text{ satisfying} \\ \sum_{i \in M} x^i_f + \sum_{j \in N} y^j_f \;=\; \sum_{i \in M} w^i_f \quad \text{for } i = 1,\dots,m. \end{array} \tag{13}$$

One can easily verifies that (r,x,y,η) satisfies (a), (d), and (e). We shall show that (b) and (c) hold.

Since $r_k > a_k$ for all $k = 1,\dots,f-1$ by (J), and $r_k = a_k$ for all $k = f,\dots,s$ by (12), it holds that for all $i \in M$, $u^i(x^i) + r(w^i - x^i) = \max_{z \in X^i}(u^i(z) + r(w^i - z))$.

Let j be a household such that $G(k-1) < j \leq G(k)$. By Lemma 2.4, if $r_t \leq I_j$ then $(e^k, I_j - r_k, \gamma_k(v)) \succeq (e^t, I_j - r_t, \gamma_t(v))$ for all $t = 1,\dots,s+1$. □

3 Tarski's Fixed-Point Theorem

Our goal of this section is to provide a proof of Lemma 2.2 using Tarski's fixed-point theorem. First, we define some basic notions of lattice theory. (For the fundamentals of lattice theory, see e.g. Davey and Priestly [1].) By a *lattice* we mean a system $L = \langle A, \leq \rangle$ formed by a nonempty set A and a binary relation $\leq$ such that $\leq$ is a partial order and that any two elements $a, b \in A$ have the least upper bound $a \vee b$ and the greatest lower bound $a \wedge b$. A lattice $L = \langle A, \leq \rangle$ is called *complete* if every subset B of A has the least upper bound $\bigvee B$ and the greatest lower bound $\bigwedge B$. For example, it is easy to see that the ordering of real numbers $\langle \mathbb{R}, \leq \rangle$ is a lattice, but not a complete one.

Let $L = \langle A, \leq \rangle$ be a lattice. Given any two elements $a, b \in A$ with $a \leq b$, we denote by $[a,b]$ the *interval* with the endpoints a and b, that is, $[a,b] = \{ x \in A \mid a \leq x \leq b \}$. The induced subsystem $\langle [a,b], \leq \rangle$ is clearly a lattice. For example, let $a, b \in \mathbb{R}$ with $a \leq b$, and let $\langle [a,b], \leq \rangle$ be the induced subsystem of $\langle \mathbb{R}, \leq \rangle$. Then $\langle [a,b], \leq \rangle$ is a complete lattice.

Let f be a function from A to A. We say that f is *monotonically increasing* if for all $x, y \in A$, $x \leq y$ implies $f(x) \leq f(y)$. The following fixed-point theorem of Tarski [5] is one of the most important results on complete lattices:

Proposition 3.1 (Tarski's fixed-point theorem) *Let $L = \langle A, \leq \rangle$ be a complete lattice, and let f be a monotonically increasing function from A to A. Then f has a fixed point, i.e. there exists an $s \in A$ such that $f(s) = s$.*

Proof. Let

$$u = \bigvee \{ x \in A \mid x \leq f(x) \}. \tag{14}$$

Let x be an element with $x \leq f(x)$. Then obviously $x \leq u$. Since f is monotonically increasing, we have $f(x) \leq f(u)$, and therefore $x \leq f(u)$. This means that $f(u)$ is an upper bound of the set $\{ x \in A \mid x \leq f(x) \}$. Hence, by (14), we have

$$u \leq f(u). \tag{15}$$

Therefore $f(u) \leq f(f(u))$, and hence $f(u)$ belongs to the set $\{ x \in A \mid x \leq f(x) \}$. By (14), we have

$$f(u) \leq u. \tag{16}$$

By (15) and (16), we conclude that u is a fixed-point of f. □

The next proposition is derived as a consequence of Tarski's fixed-point theorem.

Proposition 3.2 *Let $L = \langle A, \leq \rangle$ be a complete lattice, and let f, g be functions from $A \times A$ to A. Suppose that for all $x_1, x_2, y_1, y_2 \in A$,*

$$\begin{aligned} x_1 \leq x_2 \text{ and } y_1 \geq y_2 \quad &\text{implies} \quad f(x_1, y_1) \leq f(x_2, y_2); \\ x_1 \leq x_2 \text{ and } y_1 \geq y_2 \quad &\text{implies} \quad g(x_1, y_1) \geq g(x_2, y_2). \end{aligned} \tag{17}$$

Then there exist $s, t \in A$ such that $(f(s,t), g(s,t)) = (s,t)$.

Proof. Define a binary relation $\preceq$ on $A \times A$ by

$$(x_1, y_1) \preceq (x_2, y_2) \quad \Longleftrightarrow \quad x_1 \leq x_2 \text{ and } y_1 \geq y_2$$

We shall show that $\langle A \times A, \preceq \rangle$ is a complete lattice. It is clear that $\preceq$ is a partial order. Let S be a subset of $A \times A$. Let $S_1 = \{x \in A \mid (x,y) \in S\}$ and $S_2 = \{y \in A \mid (x,y) \in S\}$. Then, one can easily verifies that $(\bigvee S_1, \bigwedge S_2)$ is the least upper bound of S, and that $(\bigwedge S_1, \bigvee S_2)$ is the greatest lower bound of S.

We define a function $f \times g$ from $A \times A$ to $A \times A$ by $(f \times g)(x,y) = (f(x,y), g(x,y))$ for all $x, y \in A$. Then, by the condition (17), $f \times g$ is monotonically increasing with respect to $\preceq$ on $A \times A$. Hence, by Proposition 3.1, there exists $(s,t) \in A \times A$ such that $(f \times g)(s,t) = (s,t)$. □

Now we are in a position to give a proof of Lemma 2.2 in Section 2.

Proof of Lemma 2.2. Suppose there are two housing areas in the market, and let S_I, S_{II} denote the areas. We shall show that there exists a permutation π on $\{1,2,\ldots,s\}$ such that

$$(e^{\pi(1)}, 0, \gamma_{\pi(1)}(v)) \succeq (e^{\pi(2)}, 0, \gamma_{\pi(2)}(v)) \succeq \cdots \succeq (e^{\pi(s)}, 0, \gamma_{\pi(s)}(v)), \tag{18}$$

where v is the allocation of households defined by (5). Once the existence of such a permutation π is confirmed, then the conclusion (6) of Lemma 2.2 is easily obtained from (18) and Lemma 2.1(iii).

Let $\eta_I, \eta_{II} \in [0, I_1]$. We regard η_I and η_{II} as *imaginary* average incomes of S_I and S_{II}, respectively. (Here the term "imaginary" means that η_I and η_{II} are arbitrary.) Let

$$\eta_h = \begin{cases} \eta_I & \text{if } e^h \in S_I, \\ \eta_{II} & \text{if } e^h \in S_{II}, \end{cases} \tag{19}$$

for each $h = 1, \ldots, s$. It is clear from the assmption (C) that there is a permutation π on $\{1,2,\ldots,s\}$ such that

$$(e^{\pi(1)}, 0, \eta_{\pi(1)}) \succeq (e^{\pi(2)}, 0, \eta_{\pi(2)}) \succeq \cdots \succeq (e^{\pi(s)}, 0, \eta_{\pi(s)}). \tag{20}$$

Among such permutations π on $\{1,2,\ldots,s\}$ we can choose a unique permutation π such that

$$\text{for any } i,j \in \{1,2,\ldots,s\} \text{ with } i<j, \text{ if } (e^{\pi(i)}, 0, \eta_{\pi(i)}) \sim (e^{\pi(j)}, 0, \eta_{\pi(j)}) \text{ then } \pi(i) < \pi(j). \tag{21}$$

Under the permutation π, we define a function $G_\pi(k)$ $(k = 1, \ldots, s)$ by (4). Let $v = (v^{1'}, \ldots, v^{n'}) \in \{0, e^1, e^2, \ldots, e^s\}^n$ be chosen by (5). Then, the average incomes $\gamma_1(v), \ldots, \gamma_s(v)$ under the allocation v are calculated by (3). Therefore the *actual* average incomes in the areas S_I, S_{II} are determined, which we denote by γ_I, γ_{II}, respectively, that is, for all $h = 1, 2, \ldots, s$,

$$\begin{aligned} \gamma_I &= \gamma_h(v) \quad \text{if } e^h \in S_I; \\ \gamma_{II} &= \gamma_h(v) \quad \text{if } e^h \in S_{II}. \end{aligned} \tag{22}$$

According to the above argument, for a pair (η_I, η_{II}) of imaginary average incomes, we obtain the actual average incomes γ_I, γ_{II} in the areas S_I, S_{II}, respectively, under the allocation v of households. This means that we can define two functions f, g from $[0, I_1] \times [0, I_1]$ to $[0, I_1]$ by

$$f(\eta_I, \eta_{II}) = \gamma_I, \qquad g(\eta_I, \eta_{II}) = \gamma_{II}. \tag{23}$$

We shall show that the functions f, g satisfy the condition (17). It is sufficient to show that for all $\eta_I, \eta_I', \eta_{II}, \eta_{II}' \in [0, I_1]$,

$$\begin{aligned} \eta_I \leq \eta_I' &\quad \text{implies} \quad f(\eta_I, \eta_{II}) \leq f(\eta_I', \eta_{II}); && (24)\\ \eta_I \leq \eta_I' &\quad \text{implies} \quad g(\eta_I, \eta_{II}) \geq g(\eta_I', \eta_{II}); && (25)\\ \eta_{II} \geq \eta_{II}' &\quad \text{implies} \quad f(\eta_I, \eta_{II}) \leq f(\eta_I, \eta_{II}'); && (26)\\ \eta_{II} \geq \eta_{II}' &\quad \text{implies} \quad g(\eta_I, \eta_{II}) \geq g(\eta_I, \eta_{II}'). && (27) \end{aligned}$$

First, we check the conditions (24) and (25). Let $\eta_I, \eta_I', \eta_{II} \in [0, I_1]$ with $\eta_I \leq \eta_I'$. Define $\eta_1, \ldots, \eta_s$ by (19); define $\bar{\eta}_1, \ldots, \bar{\eta}_s$ by

$$\bar{\eta}_h = \begin{cases} \eta_I' & \text{if} \quad e^h \in S_I, \\ \eta_{II} & \text{if} \quad e^h \in S_{II}. \end{cases} \tag{28}$$

Let π be the permutation on $\{1, 2, \ldots, s\}$ defined by (20) and (21) with respect to $\eta_1, \ldots, \eta_s$, and let $\bar{\pi}$ be the permutation with respect to $\bar{\eta}_1, \ldots, \bar{\eta}_s$. Then, for each $e^k \in S_{II}$,

$$(e^h, 0, \eta_h) \succeq (e^k, 0, \eta_k) \quad \text{implies} \quad (e^h, 0, \bar{\eta}_h) \succeq (e^k, 0, \bar{\eta}_k) \tag{29}$$

for all $h = 1, \ldots, s$: because, if $e^h \in S_{II}$, this is obvious; if $e^h \in S_I$, then $(e^h, 0, \eta_h) \succeq (e^k, 0, \eta_k)$ implies $(e^h, 0, \bar{\eta}_h) \succeq (e^h, 0, \eta_h) \succeq (e^k, 0, \eta_k) \sim (e^k, 0, \bar{\eta}_k)$ by (E). This means that, for each $e^k \in S_{II}$, $\pi(i) = \bar{\pi}(j) = k$ implies $i \leq j$. Let $v = (v^1, \ldots, v^n)$ be the allocation of households defined by (5) with respect to G_π, and let $\bar{v} = (\bar{v}^1, \ldots, \bar{v}^n)$ be defined by (5) with respect to $G_{\bar{\pi}}$. Then, by the definitions of v and $\bar{v}$, it is easy to see that for each $e^k \in S_{II}$, the incomes of the households in the kth apartment under the allocation $\bar{v}$ are not higher than that under the allocation v. Therefore, for $e^k \in S_{II}$,

$$\sum_{v^j \in S(k)} I_j \geq \sum_{\bar{v}^j \in S(k)} I_j. \tag{30}$$

Let us divide both sides of (30) by $\sum_{i \in M} \sum_{e^h \in S(k)} w_h^i$. Then we obtain $\gamma_k(v) \geq \gamma_k(\bar{v})$ for $e^k \in S_{II}$. Thus the condition (25) is proved. We also deduce from (30) that for $e^l \in S_I$, $\sum_{v^j \in S(l)} I_j \leq \sum_{\bar{v}^j \in S(l)} I_j$. Hence $\gamma_l(v) \leq \gamma_l(\bar{v})$ for $e^l \in S_I$, and thus the condition (24) is proved. The conditions (26) and (27) can also be proved in parallel ways to (25) and (24), respectively.

Therefore the functions f, g satisfy the condition (17). Hence, by Proposition 3.2 and (23), there exist η_I, η_{II} such that $\eta_I = \gamma_I$ and $\eta_{II} = \gamma_{II}$. Then, by (19) and (22), for each $h = 1, 2, \ldots, s$,

$$\text{if } e^h \in S_I \text{ then } \eta_h = \eta_I = \gamma_I = \gamma_h(v); \quad \text{if } e^h \in S_{II} \text{ then } \eta_h = \eta_{II} = \gamma_{II} = \gamma_h(v).$$

This means $(\eta_1, \ldots, \eta_s) = (\gamma_1(v), \ldots, \gamma_s(v))$. Therefore the condition (20) yields the conclusion (18).

This completes the proof of Lemma 2.2.

4 Concluding Remarks

We presented a model of rental housing market with externalities, where the externalities among households are considered in such a way that the average income of households has an effect on a household's consumption behavior. We proved the existence of a competitive equilibrium in our model using Tarski's fixed-point theorem, under the assumption that there are only two housing areas.

Our future work is to generalize Lemma 2.2 to the case of three or more areas, and to prove Theorem 2.1 without the assumption (H). In the case of three areas, however, Tarski's fixed-point theorem is not useful to prove an analogue of Lemma 2.2, and hence we will need to prove it in another way, possibly with additional assumptions.

Appendix

Proof of Lemma 2.1. (i) Suppose $(y,m_2-\delta,\eta_2) \succeq (x,m_1-\delta,\eta_1)$. By (D) and (F), there is a $b \geq 0$ such that $(y,m_2-\delta,\eta_2) \sim (x,m_1-\delta+b,\eta_1)$. Since $m_1-\delta+b > m_2-\delta$, we have $(y,m_2,\eta_2) \succ (x,m_1+b,\eta_1)$ by (G). But, by (D) we have $(x,m_1+b,\eta_1) \succeq (x,m_1,\eta_1) \sim (y,m_2,\eta_2)$, that is, $(x,m_1+b,\eta_1) \succeq (y,m_2,\eta_2)$, a contradiction.

(ii) Suppose $m_1 > m_2$. By (D), we have $(y,m_1,\eta_2) \succ (y,m_2,\eta_2)$. Therefore $(x,m_1,\eta_1) \succeq (y,m_1,\eta_2) \succ (y,m_2,\eta_2)$, that is, $(x,m_1,\eta_1) \succ (y,m_2,\eta_2)$. This contradicts to $(x,m_1,\eta_1) \sim (y,m_2,\eta_2)$.

(iii) By (D) and (F), there is a $b>0$ such that $(x,m_1,\eta_1) \sim (y,m_1+b,\eta_2)$. By (G), we have $(x,m_2,\eta_1) \succ (y,m_2+b,\eta_2)$. Hence $(x,m_2,\eta_1) \succ (y,m_2+b,\eta_2) \succ (y,m_2,\eta_2)$.

Proof of Lemma 2.3. (i) Since $(e^f, I_{G(f-1)}-a_f, \gamma_f(v)) \succeq (e^f, I_n-a_f, \gamma_f(v)) \succeq (e^1, 0, \gamma_1(v)) \succeq \cdots \succeq (e^{f-1}, 0, \gamma_{f-1}(v))$ by (D), (7), and the supposition of this lemma, there is a b_{f-1} such that $(e^f, I_{G(f-1)}-a_f, \gamma_f(v)) \sim (e^{f-1}, b_{f-1}, \gamma_{f-1}(v))$. This b_{f-1} is unique by (D). Let $p_{f-1} = I_{G(f-1)} - b_{f-1}$. Since $(e^{f-1}, I_{G(f-2)}-p_{f-1}, \gamma_{f-1}(v)) \succeq (e^{f-1}, I_{G(f-1)}-p_{f-1}, \gamma_{f-1}(v)) \sim (e^f, I_{G(f-1)}-a_f, \gamma_f(v)) \succeq (e^1, 0, \gamma_1(v)) \succeq \cdots \succeq (e^{f-2}, 0, \gamma_{f-2}(v))$ by (D), (7), and the supposition of this lemma, there is a b_{f-2} such that $(e^{f-1}, I_{G(f-2)}-p_{f-1}, \gamma_{f-1}(v)) \sim (e^{f-2}, b_{f-2}, \gamma_{f-2}(v))$. Let $p_{f-2} = I_{G(f-2)} - b_{f-2}$. Repeating this argument we get $(p_1, \ldots, p_{f-1})$. Lemma 2.1(ii) implies that if $(e^k, I_{G(k)}-p_k, \gamma_k(v)) \sim (e^{k+1}, I_{G(k)}-p_{k+1}, \gamma_{k+1}(v))$, then $I_{G(k)}-p_k \leq I_{G(k)}-p_{k+1}$, that is, $p_k \geq p_{k+1}$.

Proof of Lemma 2.4. (i) Let $k \leq t \leq f-1$. Since $(e^t, I_{G(t)}-p_t, \gamma_t(v)) \sim (e^{t+1}, I_{G(t)}-p_{t+1}, \gamma_{t+1}(v))$ and $p_t \geq p_{t+1}$, we have $(e^t, I-p_t, \gamma_t(v)) \succeq (e^{t+1}, I-p_{t+1}, \gamma_{t+1}(v))$ by (G) or (7). Hence the first proposition is proved. Let $t>f$ with $I_n \geq a_t$. Since $(e^f, I_n-a_f, \gamma_f(v)) \succ (e^t, I_n-a_t, \gamma_t(v))$ by (I), there is a $b_t>0$ such that $(e^f, I_n-a_f, \gamma_f(v)) \sim (e^t, I_n-a_t+b_t, \gamma_t(v))$ by (D) and (F). By Lemma 2.1(ii), we have $I_n-a_f \leq I_n-a_t+b_t$. Hence, $(e^f, I-a_f, \gamma_f(v)) \succeq (e^t, I-a_t+b_t, \gamma_t(v))$ by (G) or (7). Therefore $(e^f, I-a_f, \gamma_f(v)) \succeq (e^t, I-a_t+b_t, \gamma_t(v)) \succ (e^t, I-a_t, \gamma_t(v))$. Let $t>f$ with $I \geq a_t > I_n$. Since $(e^f, 0, \gamma_f(v)) \succeq (e^t, 0, \gamma_t(v))$ by (7), there is a $b_t \geq 0$ such that $(e^f, 0, \gamma_f(v)) \sim (e^t, b_t, \gamma_t(v))$. Hence, $(e^f, I-a_t, \gamma_f(v)) \succeq (e^t, I-a_t+b_t, \gamma_t(v))$ by (G) or (7). Therefore $(e^f, I-a_t, \gamma_f(v)) \succeq (e^t, I-a_t+b_t, \gamma_t(v)) \succeq (e^t, I-a_t, \gamma_t(v))$. Since $I_n-a_f \geq 0 > I_n-a_t$, we have $I-a_f > I-a_t$, which implies $(e^f, I-a_f, \gamma_f(v)) \succ (e^f, I-a_t, \gamma_f(v))$. Thus $(e^f, I-a_f, \gamma_f(v)) \succ (e^t, I-a_t, \gamma_t(v))$.

(ii) Let $t<k$ with $p_t \leq I$. Suppose $(e^t, I-p_t, \gamma_t(v)) \succ (e^{t+1}, I-p_{t+1}, \gamma_{t+1}(v))$. By (D) and (F), there is a $b>0$ such that $(e^t, I-p_t, \gamma_t(v)) \sim (e^{t+1}, I-p_{t+1}+b, \gamma_{t+1}(v))$. Since $p_t \geq p_{t+1}$, we have $I-p_t < I-p_{t+1}+b$. Therefore $(e^t, I_{G(t)}-p_t, \gamma_t(v)) \succeq (e^{t+1}, I_{G(t)}-p_{t+1}+b, \gamma_{t+1}(v))$ by (G). Hence $(e^t, I_{G(t)}-p_t, \gamma_t(v)) \succeq (e^{t+1}, I_{G(t)}-p_{t+1}+b, \gamma_{t+1}(v)) \succ (e^{t+1}, I_{G(t)}-p_{t+1}, \gamma_{t+1}(v))$. But this contradicts to $(e^t, I_{G(t)}-p_t, \gamma_t(v)) \sim (e^{t+1}, I_{G(t)}-p_{t+1}, \gamma_{t+1}(v))$. Therefore $(e^{t+1}, I-$

$p_{t+1}, \gamma_{t+1}(v)) \succeq (e^t, I - p_t, \gamma_t(v))$.

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An Approach to Housing Markets with Externalities using Tarski's Fixed-Point Theorem

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This paper presents a model of rental housing markets with externalities, in which households and landlords trade apartments as indivisible commodities. The externalities among households are considered in such a way that the average income of households affects on a household's consumption behavior. The existence of a competitive equilibrium in this model is proved using Tarski's fixed-point theorem.

Key Words : rental housing markets, externalities, competitive equilibria, Tarski's fixed-point theorem